

Paradoxes in Social Networks with Multiple Products

Krzysztof R. Apt

*Centre for Mathematics and Computer Science (CWI)
and University of Amsterdam, The Netherlands*

Evangelos Markakis

Athens University of Economics and Business, Athens, Greece

Sunil Simon

Centre for Mathematics and Computer Science (CWI)

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Abstract

Recently, we introduced in [1] a model for product adoption in social networks with multiple products, where the agents, influenced by their neighbours, can adopt one out of several alternatives. We identify and analyze here four types of paradoxes that can arise in these networks. To this end, we use social network games that we recently introduced in [15]. These paradoxes shed light on possible inefficiencies arising when one modifies the sets of products available to the agents forming a social network. One of the paradoxes corresponds to the well-known Braess paradox in congestion games and shows that by adding more choices to a node, the network may end up in a situation that is (weakly) worse for everybody. We exhibit a dual version of this, where removing available choices from someone can eventually make everybody better off. The other paradoxes that we identify show that by adding or removing a product from the choice set of some node may lead to permanent instability. Finally, we also identify conditions under which some of these paradoxes cannot arise.

1 Introduction

One of the most striking paradoxes in game theory is Braess paradox. It states that in some road networks the travel time can actually increase when new roads are added, see, e.g., [14, pages 464-465]. This paradox can be expressed as a statement about the congestion games, in which the players' objective is to minimize the travel time from the source to the sink, and the cost (travel time) depends negatively on the number of users of each road segment. Then Braess

paradox states that in some congestion games an addition of a new strategy (road segment) can trigger a sequence of changes (an improvement path) that brings the players from the initial Nash equilibrium to a new one with worse costs (travel time) for each player.

Braess paradox has received a great deal of attention because of its counterintuitive character and potential implications. It has been studied in other contexts too, for example queueing networks, see [3]. A natural ‘dual’ version of this paradox, concerning the removal of road segments, has also been studied, see [5, 6]. This version states that in some congestion games a removal of a strategy (road segment) can trigger a sequence of changes (an improvement path) that brings the players from the initial Nash equilibrium to a new one with better costs (travel time) for each player.

The focus of this paper is social networks. This has become a huge interdisciplinary research area with important links to sociology, economics, epidemiology, computer science, and mathematics. A flurry of numerous articles, notably the influential [13], and books, see [2, 8, 17, 9, 4], helped to delineate better this area. It deals with such diverse topics as epidemics, spread of certain patterns of social behaviour, effects of advertising, and emergence of ‘bubbles’ in financial markets.

Recently, we introduced in [1] *social networks with multiple products*, in which the agents (players), influenced by their neighbours, can adopt one out of several alternatives. An example of such a network is a group of people who choose providers of mobile phones by taking into account the choice of their friends. To analyze the dynamics of such networks we introduced in [15] and more fully in [16] a natural class of *social network games*. In these strategic games the payoff of each player weakly increases when more players choose the same product (strategy) as him - exactly the opposite of what happens in congestion games.

The purpose of this paper is to show that in these social network games, paradoxes similar to Braess paradox and its dual version exist. These paradoxes provide us with insights into the possible changes triggered by an addition or removal of products in the considered social networks. We show that in addition to the above two paradoxes two other types of paradoxes exist. In particular, it is possible that an addition of a new product to (respectively, a removal of a product from) the choice set of a player results in a permanent instability, in the sense that the resulting game has no Nash equilibrium.

We also analyze variants of these paradoxes that are for example obtained by stipulating that the corresponding ‘new situation’ is inevitable instead of only being possible.

The general setup in which we study these paradoxes makes it possible to interpret them as phenomena that can take place in any community the members of which make choices by taking into account the choices of others. An example is a ‘bubble’ in a financial market, where a decision of a trader to switch to some new financial product triggers a sequence of transactions, as a result of which all traders involved become worse off.

Further, it was noticed in a number of empirical studies that an abundance

of choices may sometimes lead to wrong decisions. To quote from [7, page 38]:

The freedom-of-choice paradox. The more options one has, the more possibilities for experiencing conflict arise, and the more difficult it becomes to compare the options. There is a point where more options, products, and choices hurt both seller and consumer.

Both phenomena can be naturally explained in our framework.

The paper is organized as follows. In the next section we introduce the background material. In Sections 3, 4, 5, and 6, using the social network games, we define formally and analyze four types of paradoxes. Then, in Section 7 we consider the case of networks, where the underlying graph has no source nodes. We also study simple cycles as a special case of this. Finally, in Section 8 we discuss future research directions.

2 Preliminaries

2.1 Strategic games

A **strategic game** for $n > 1$ players, written as $(S_1, \dots, S_n, p_1, \dots, p_n)$, consists of a non-empty set S_i of **strategies** and a **payoff function** $p_i : S_1 \times \dots \times S_n \rightarrow \mathbb{R}$, for each player i .

Fix a strategic game $G := (S_1, \dots, S_n, p_1, \dots, p_n)$. We denote $S_1 \times \dots \times S_n$ by S , call each element $s \in S$ a **joint strategy**, denote the i th element of s by s_i , and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i} . Occasionally we write (s_i, s_{-i}) instead of s .

We call a strategy s_i of player i a **best response** to a joint strategy s_{-i} of his opponents if $\forall s'_i \in S_i p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$. We call a joint strategy s a **Nash equilibrium** if each s_i is a best response to s_{-i} . Further, we call a strategy s'_i of player i a **better response** given a joint strategy s if $p_i(s'_i, s_{-i}) > p_i(s_i, s_{-i})$.

By a **profitable deviation** we mean a pair (s, s') of joint strategies such that $s' = (s'_i, s_{-i})$ for some s'_i and $p_i(s') > p_i(s)$. Further, when s'_i is a best response to s_{-i} , we call it a **best response deviation**. Following [12], an **improvement path** (respectively, a **best response improvement path**) is a maximal sequence of profitable deviations (respectively, best response deviations). Clearly, if a (best response) improvement path is finite, then its last element is a Nash equilibrium.

Given two joint strategies s and s' we write

- $s >_w s'$ if for all i , $p_i(s) \geq p_i(s')$ and for some i , $p_i(s) > p_i(s')$,
- $s >_s s'$ if for all i , $p_i(s) > p_i(s')$.

When $s >_w s'$ (respectively, $s >_s s'$) holds we say that s' is **weakly worse** (respectively, **strictly worse**) than s .

2.2 Social networks

We are interested in strategic games defined over a specific type of social networks recently introduced in [1] that we recall first.

Let $V = \{1, \dots, n\}$ be a finite set of *agents* and $G = (V, E, w)$ a weighted directed graph with $w_{ij} \in [0, 1]$ being the weight of the edge (i, j) . Given a node i of G , we denote by $N(i)$ the set of nodes from which there is an incoming edge to i . We call each $j \in N(i)$ a *neighbour* of i in G . We assume that for each node i such that $N(i) \neq \emptyset$, $\sum_{j \in N(i)} w_{ji} \leq 1$. An agent $i \in V$ is said to be a *source node* in G if $N(i) = \emptyset$. Given a (to be defined) network \mathcal{S} we denote by $\text{source}(\mathcal{S})$ the set of source nodes in the underlying graph G .

By a *social network* (from now on, just *network*) we mean a tuple $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, where

- G is a weighted directed graph,
- \mathcal{P} is a finite set of alternatives or *products*,
- P is function that assigns to each agent i a non-empty set of products $P(i)$ from which it can make a choice,
- θ is a *threshold function* that for each $i \in V$ and $t \in P(i)$ yields a value $\theta(i, t) \in (0, 1]$.

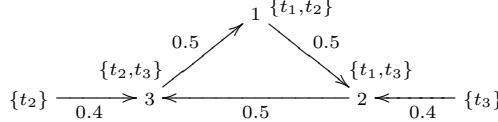


Figure 1: A social network

Example 1. Figure 1 shows an example of a network. Let the threshold be 0.3 for all nodes. The set of products \mathcal{P} is $\{t_1, t_2, t_3\}$, the product set of each agent is marked next to the node denoting it and the weights are labels on the edges. Each source node is represented by the unique product in its product set. \square

Given two social networks \mathcal{S} and \mathcal{S}' we say that \mathcal{S}' is an *expansion* of \mathcal{S} if it results from adding a product to the product set of a node in \mathcal{S} . We say then also that \mathcal{S} is a *contraction* of \mathcal{S}' .

2.3 Social network games

Next, we recall the strategic games over the social networks in the above sense that we introduced in [15]. Fix a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$. Each agent can adopt a product from his product set or choose not to adopt any product. We denote the latter choice by t_0 .

With each network \mathcal{S} we associate a strategic game $\mathcal{G}(\mathcal{S})$. The idea is that the agents simultaneously choose a product or abstain from choosing any. Subsequently each node assesses his choice by comparing it with the choices made by his neighbours. Formally, we define the game as follows:

- the players are the agents (i.e., the nodes),
- the set of strategies for player i is $S_i := P(i) \cup \{t_0\}$,
- For $i \in V$, $t \in P(i)$ and a joint strategy s , let $\mathcal{N}_i^t(s) := \{j \in N(i) \mid s_j = t\}$, i.e., $\mathcal{N}_i^t(s)$ is the set of neighbours of i who adopted in s the product t .

The payoff function is defined as follows, where c_0 is some given in advance positive constant:

$$\begin{aligned}
 & \text{– for } i \in \text{source}(\mathcal{S}), \\
 p_i(s) &:= \begin{cases} 0 & \text{if } s_i = t_0 \\ c_0 & \text{if } s_i \in P(i) \end{cases} \\
 & \text{– for } i \notin \text{source}(\mathcal{S}), \\
 p_i(s) &:= \begin{cases} 0 & \text{if } s_i = t_0 \\ \sum_{j \in \mathcal{N}_i^t(s)} w_{ji} - \theta(i, t) & \text{if } s_i = t, \text{ for some } t \in P(i) \end{cases}
 \end{aligned}$$

In the first entry we assume that the payoff function for the source nodes is constant only for simplicity. The second entry in the payoff definition is motivated by the following considerations. When agent i is not a source node, his ‘satisfaction’ from a joint strategy depends positively from the accumulated weight (read: ‘influence’) of his neighbours who made the same choice as him, and negatively from his threshold level (read: ‘resistance’) to adopt this product. The assumption that $\theta(i, t) > 0$ reflects the view that there is always some resistance to adopt a product. Strategy t_0 represents the possibility that an agent refrains from choosing a product.

Example 2. Consider the network given in Example 1 and the joint strategy s where each source node chooses the unique product in its product set and nodes 1, 2 and 3 choose t_2 , t_3 and t_2 respectively. The payoffs are then given as follows:

- for the source nodes, the payoff is the fixed constant c_0 ,
- $p_1(s) = 0.5 - 0.3 = 0.2$,
- $p_2(s) = 0.4 - 0.3 = 0.1$,
- $p_3(s) = 0.4 - 0.3 = 0.1$.

Let s' be the joint strategy in which player 3 chooses t_3 and the remaining players make the same choice as given in s . Then (s, s') is a profitable deviation since $p_3(s') > p_3(s)$. In what follows, we represent each profitable deviation by a node and a strategy it switches to, e.g., $3 : t_3$. Starting at s , the sequence of profitable deviations $3 : t_3, 1 : t_0$ is an improvement path which results in the joint strategy in which nodes 1, 2 and 3 choose t_0, t_3 and t_3 respectively and each source node chooses the unique product in its product set. \square

By definition, the payoff of each player depends only on the strategies chosen by his neighbours, so the social network games are related to graphical games of [10]. However, the underlying dependence structure of a social network game is a directed graph and the presence of the special strategy t_0 available to each player makes these games more specific. Finally, note that these games satisfy the ***join the crowd*** property that we define as follows:

Each payoff function p_i depends only on the strategy chosen by player i and the set of players who also chose his strategy. Moreover, the dependence on this set is monotonic.

The last qualification is exactly opposite to the definition of congestion games with player-specific payoff functions of [11] in which the dependence on the above set is antimonotonic. That is, when more players choose the strategy of player i , then his payoff weakly decreases.

3 Vulnerable networks

In what follows we introduce and analyze four types of deficient networks. In this section we focus on the following notions.

We say that a social network \mathcal{S} is $\exists w$ -**vulnerable** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, an expansion \mathcal{S}' of \mathcal{S} exists such that some improvement path in $\mathcal{G}(\mathcal{S}')$ leads from s to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$ such that $s >_w s'$. In general we have four notions of vulnerability, that correspond to the combinations XY , where $X \in \{\exists, \forall\}$ and $Y \in \{w, s\}$. For example, we say that \mathcal{S} is $\forall s$ -**vulnerable** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, an expansion \mathcal{S}' of \mathcal{S} exists such that each improvement path in $\mathcal{G}(\mathcal{S}')$ leads from s to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S})$ such that $s >_s s'$.

First note that there are some obvious implications between the four notions of vulnerability and inefficiency that we exhibit in Figure 2.

We show now that these implications are the only ones that hold between these four notions.

Example 3 ($\forall w$). In Figure 3 we exhibit an example of a $\forall w$ -vulnerable network that is not $\exists s$ -vulnerable. The product set of each node is marked next to it and the weights are labels on the edges. We assume that each threshold is a constant θ , where $0 < \theta < 0.1$. Here and elsewhere the relevant expansion is depicted by means of a product and the dotted arrow pointing to the relevant node. In this case product t_1 is added to node 4.

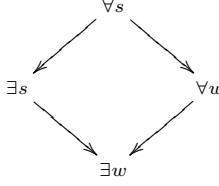


Figure 2: Dependencies between the notions of vulnerability

The initial Nash equilibrium s is the joint strategy formed by the underlined products, i.e., $(t_2, t_3, t_3, t_3, t_1, t_1)$. Consider now what happens after product t_1 is added to the product set of node 4. Then s ceases to be a Nash equilibrium. Addition of t_1 triggers the unique best response improvement path

$$4 : t_1, 3 : t_2, 5 : t_2, 6 : t_0, 4 : t_3, 3 : t_3, 5 : t_0$$

resulting in the Nash equilibrium $(t_2, t_3, t_3, t_3, t_0, t_0)$. Note that at each step of any improvement path starting in s triggered by the addition of product t_1 to node 4 there is a unique node which is not playing its best response. For instance, in the second step of the above improvement path, node 3 is the unique node which is not playing its best response. Although node 3 can profitably deviate to t_0 instead of t_2 , in the next step which is unique, node 3 is forced to play its best response. Therefore it suffices to consider the outcome of the above best response improvement path. In the Nash equilibrium $(t_2, t_3, t_3, t_3, t_0, t_0)$ the payoffs of players 1–4 did not change with respect to the original Nash equilibrium, while the payoffs of players 5 and 6 decreased.

Finally, notice that a network is not $\exists s$ -vulnerable if the underlying graph has a source node. \square

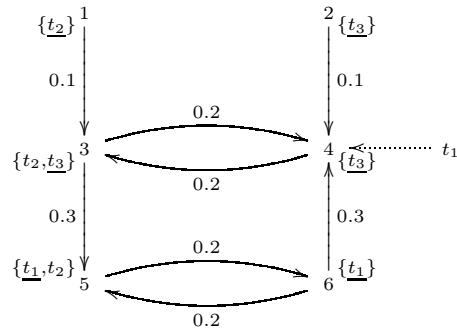


Figure 3: An $\forall w$ -vulnerable network

In this specific example the payoff of the player who triggered the change in the end did not change. A slightly more complicated example, that we omit, shows that the initiator's payoff in the final Nash equilibrium can decrease. Also,

one can construct examples in which the payoffs in the final Nash equilibrium decrease for an arbitrary large fraction of the players and remain constant for the other players.

Example 4 ($\exists s$). In Figure 4 we exhibit an example of a $\exists s$ -vulnerable network that is not $\forall w$ -vulnerable (and hence not $\forall s$ -vulnerable). As before we assume that each threshold is a constant θ , where $0 < \theta < 0.1$ and we underline the strategies that form the initial Nash equilibrium.

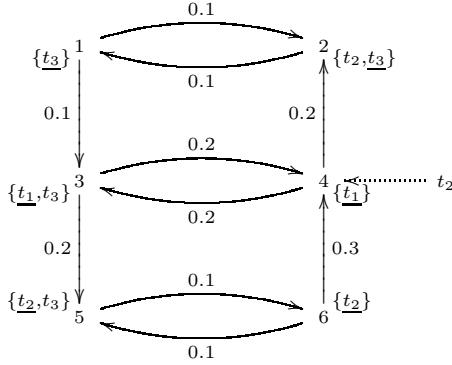


Figure 4: A $\exists s$ -vulnerable network

To see that this network is $\exists s$ -vulnerable it suffices to note that starting from the Nash equilibrium $(t_3, t_3, t_1, t_1, t_2, t_2)$ of the initial network the addition of product t_2 to node 4 triggers the best response improvement path

$$4 : t_2, 3 : t_3, 5 : t_3, 6 : t_0, 2 : t_2, 1 : t_0, 4 : t_0, 2 : t_0, 3 : t_0, 5 : t_0$$

that ends in a Nash equilibrium in which each strategy equals t_0 , and consequently each payoff becomes 0.

To see that this network is not $\forall w$ -vulnerable first note that the addition of product t_2 to node 4 also triggers the improvement path

$$4 : t_2, 2 : t_2, 1 : t_0, 3 : t_0$$

that ends in a Nash equilibrium in which the payoffs of nodes 2 and 4 increase.

Each of the remaining initial Nash equilibria includes a strategy t_0 . But when a joint strategy includes t_0 , then for no s' we have $s >_s s'$. This allows us to conclude that the considered network is not $\forall s$ -vulnerable. To show that the considered network is in fact not $\forall w$ -vulnerable we need to analyze each of the initial Nash equilibria and consider all profitable additions. We only consider one representative example. Consider the initial Nash equilibrium $(t_0, t_0, t_1, t_1, t_2, t_2)$ and the addition of product t_1 to node 2. This triggers the unique improvement path $2 : t_1, 1 : t_0$ that ends in a Nash equilibrium in which the payoff of node 2 increased. \square

If we just wish to construct an improvement path, so not necessarily a best response improvement path, that yields a strictly worse Nash equilibrium, then a simpler example can be used. Namely, one can drop in the above network the nodes 1 and 2 and all the arcs to and from them, and adjust the threshold function of node 3 so that $\theta(3, t_3) < \theta(3, t_1)$. Then

$$4 : t_2, 3 : t_3, 5 : t_3, 6 : t_0, 4 : t_0, 3 : t_0, 5 : t_0$$

is the desired improvement path.

Example 5 ($\exists w$). Next, we provide an example of a $\exists w$ -vulnerable network that is neither $\exists s$ -vulnerable nor $\forall w$ -vulnerable. It suffices to add to the network given in Figure 4 a source node 7 with the product set $\{t_1\}$ and connect it to node 1 using an arbitrary threshold and weight. In each Nash equilibrium, node 7 chooses t_1 , so its payoff is the same. Further, the choice of this node has no influence on the choices of other nodes in the Nash equilibria in the original and the extended networks. So the conclusion follows from the previous example. \square

Next, we would like to mention the following intriguing question:

Open problem: Do $\forall s$ -vulnerable networks exist?

The following result shows that if they do, they use at least three products.

Theorem 6. *When there are only two products $\forall w$ -vulnerable networks, so a fortiori $\forall s$ -vulnerable networks, do not exist.*

Proof. Suppose by contradiction that such a network \mathcal{S} exists. So a Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, a node, say 1, and a product, say t_1 , exists such that for the network expansion \mathcal{S}' obtained by adding t_1 to the product set of node 1 each improvement path that starts in s ends up in a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$ such that $s >_w s'$.

Given an initial joint strategy we call a maximal sequence of best response deviations to a strategy t (in an arbitrary order) a **t -phase**. We now repeatedly perform, starting at s , the t_1 -phase followed by the t_0 -phase. We claim that this process terminates and hence yields a finite improvement path in $\mathcal{G}(\mathcal{S}')$.

First note that if a joint strategy s^2 is obtained from s^1 by having some nodes to switch to product t_1 and t_1 is a best response for a node i to s_{-i}^1 , then t_1 is also a best response for i to s_{-i}^2 . Indeed, by the join the crowd property $p_i(t_1, s_{-i}^2) \geq p_i(t_1, s_{-i}^1)$ and $p_i(t_2, s_{-i}^1) \geq p_i(t_2, s_{-i}^2)$, so $p_i(t_1, s_{-i}^2) \geq p_i(t_2, s_{-i}^2)$ since $p_i(t_1, s_{-i}^1) \geq p_i(t_2, s_{-i}^1)$. Further, $p_i(t_1, s_{-i}^1) \geq p_i(t_0, s_{-i}^1)$, so also $p_i(t_1, s_{-i}^2) \geq p_i(t_0, s_{-i}^2)$. Consequently after the first t_1 -phase each node that has the strategy t_1 plays a best response. Call the outcome of the first t_1 -phase s'' .

Consider now a node i that deviated in s'' to t_0 by means of a best response. By the observation just made node i deviated from product t_2 . So, again by the join the crowd property, this deviation does not affect the property that the nodes that selected t_1 in s'' play a best response. Iterating this reasoning

we conclude that after the first t_0 -phase each node that has the strategy t_1 continues to play a best response.

By the same reasoning subsequent t_1 and t_0 -phases have the same effect on the set of nodes that have the strategy t_1 : each of these nodes continues to play a best response.

Moreover, this set continues to weakly increase. Consequently these repeated applications of the t_1 -phase followed by the t_0 -phase terminate, say in a joint strategy s' . Suppose now a node i does not play a best response to s'_{-i} . If $s'_i = t_0$, then by the construction t_1 is not a best response, so t_2 is a best response.

Suppose the initial strategy of node i was also t_0 , i.e., $s_i = t_0$. Since s is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$, we have $p_i(t_2, s_{-i}) \leq p_i(t_0, s_{-i})$. By the join the crowd property $p_i(t_2, s'_{-i}) \leq p_i(t_2, s_{-i})$, so $p_i(t_2, s'_{-i}) \leq p_i(t_0, s'_{-i})$, which yields a contradiction. Hence node i deviated to t_0 from some intermediate joint strategy s^1 by selecting a best response. So $p_i(t_2, s'_{-i}) \leq p_i(t_0, s^1_{-i})$. Moreover, by the join the crowd property $p_i(t_2, s'_{-i}) \leq p_i(t_2, s^1_{-i})$, so $p_i(t_2, s'_{-i}) \leq p_i(t_0, s'_{-i})$, which yields a contradiction, as well.

Further, by the construction $s'_i \neq t_1$, so the only alternative is that $s'_i = t_2$. But then either t_0 or t_1 is a best response, which contradicts the construction of s' . We conclude that s' is a Nash equilibrium in $\mathcal{G}(\mathcal{S}')$.

Next, the payoff of node 1 strictly increased when it switched to t_1 and, on the account of the above arguments, during the remaining steps of the considered improvement path it either increased or remained the same. We conclude that the final Nash equilibrium s' is not weakly worse than the original, which yields a contradiction. \square

4 Fragile networks

Related notions to vulnerable networks are the following ones.

We say that a social network \mathcal{S} is \exists -**fragile** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, an expansion \mathcal{S}' of \mathcal{S} exists such that some improvement path in $\mathcal{G}(\mathcal{S}')$ that starts in s is infinite. In turn, we say that a social network \mathcal{S} is \forall -**fragile** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, an expansion \mathcal{S}' of \mathcal{S} exists such that each improvement path in $\mathcal{G}(\mathcal{S}')$ that starts in s is infinite. Finally, we say that a social network \mathcal{S} is **fragile** if $\mathcal{G}(\mathcal{S})$ has a Nash equilibrium, while for some expansion \mathcal{S}' of \mathcal{S} , $\mathcal{G}(\mathcal{S}')$ does not.

Obviously each fragile network is \forall -fragile, while each \forall -fragile network is \exists -fragile. We now show that these two implications are proper.

Example 7 (Fragile). Consider the network \mathcal{S} given in Figure 5 where the source nodes are represented by the unique product in their product set. We assume that each threshold is a constant θ such that $\theta < w_1 < w_2$.

Consider the joint strategy s , in which the nodes marked by $\{t_1\}$, $\{t_2\}$ and $\{t_3\}$ choose the unique product in their product set and nodes 1, 2, and 3 choose t_1 , t_1 and t_2 , respectively. For convenience, we denote s by the choices of nodes

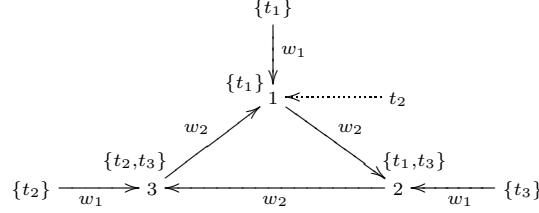


Figure 5: A fragile network

1, 2 and 3, so $s = (t_1, t_1, t_2)$. It is easy to verify that s is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$.

Consider now the expansion \mathcal{S}' of \mathcal{S} in which product t_2 is added to the product set of node 1. In $\mathcal{G}(\mathcal{S}')$ the joint strategy s ceases to remain a Nash equilibrium. In fact, no joint strategy is a Nash equilibrium in $\mathcal{G}(\mathcal{S}')$. Each agent residing on the triangle can secure a payoff of at least $w_1 - \theta > 0$, so it suffices to analyze the joint strategies in which t_0 is not used. There are in total eight such joint strategies. Here is their listing, where in each joint strategy we underline the strategy that is not a best response to the choice of other players: $(\underline{t_1}, t_1, t_2)$, $(t_1, \underline{t_1}, t_3)$, $(t_1, t_3, \underline{t_2})$, $(t_1, \underline{t_3}, t_3)$, $(t_2, \underline{t_1}, t_2)$, $(t_2, \underline{t_1}, t_3)$, $(t_2, t_3, \underline{t_2})$, $(\underline{t_2}, t_3, t_3)$. This shows that the initial network \mathcal{S} is fragile. \square

Example 8 (\forall -fragile). Consider the network \mathcal{S} given in Figure 6. We assume

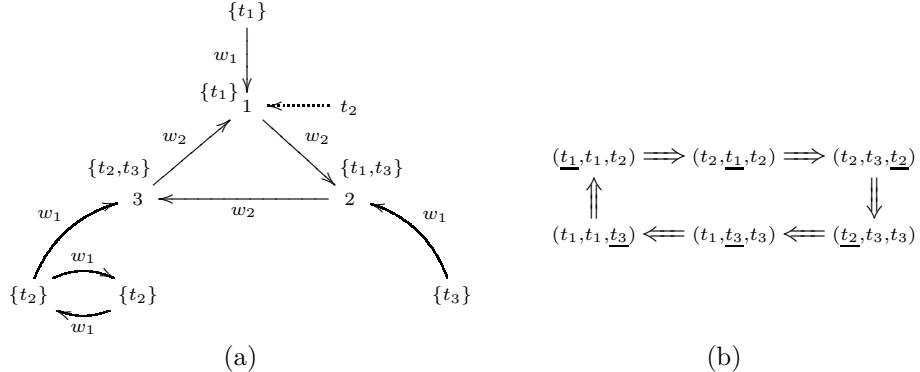


Figure 6: A \forall -fragile network and an infinite improvement path

that each threshold is a constant θ , where $\theta < w_1 < w_2$. Consider the joint strategy s , in which the nodes marked by $\{t_1\}$, $\{t_2\}$ and $\{t_3\}$ choose the unique product in their product set and nodes 1, 2, and 3 choose t_1, t_1 and t_2 , respectively. As in the previous example we denote s by (t_1, t_1, t_2) . It is easy to check that s is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$.

Now consider the expansion \mathcal{S}' obtained by adding the product t_2 to the product set of node 1. The joint strategy s ceases to remain a Nash equilibrium

in $\mathcal{G}(\mathcal{S}')$. In fact, Figure 6(b) shows the unique best response improvement path starting in s which is infinite. For each joint strategy in the figure, we underline the strategy that is not a best response. As in the case of Example 3 at every step of every improvement path starting in s , there is a unique node which is not playing its best response. Therefore it suffices to consider the above best response improvement path. This shows that \mathcal{S} is \forall -fragile.

Also note that the game $\mathcal{G}(\mathcal{S}')$ has a Nash equilibrium. The joint strategy in which the nodes marked $\{t_2\}$ along with node 3 choose the product t_0 , the node marked $\{t_3\}$ choose the product t_3 and the node marked $\{t_1\}$ along with nodes 1 and 2 choose t_1 forms a Nash equilibrium. It follows that \mathcal{S} is not fragile. \square

Example 9 (\exists -fragile). Consider the network \mathcal{S} given in Figure 7(a). Let the

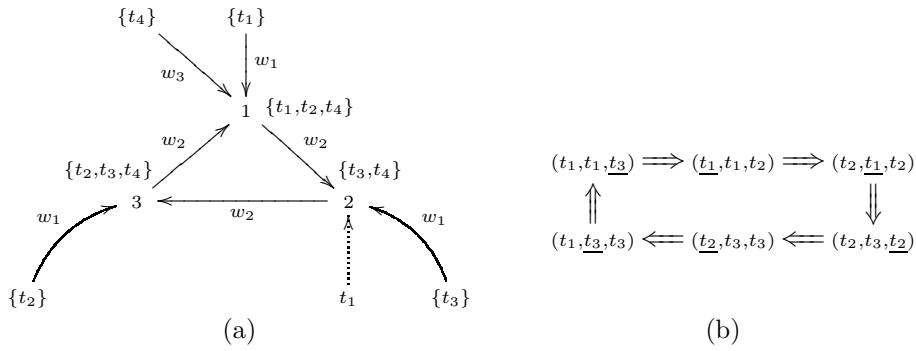


Figure 7: A \exists -fragile network and an infinite improvement path

threshold be a constant θ , where $\theta < w_3 < w_1 < w_2$. Assume that each source node selects its unique product. Identify each joint strategy that extends this selection with the selection of the strategies by the nodes 1, 2 and 3. The joint strategy $s = (t_1, t_3, t_3)$ is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$.

Now consider the expansion \mathcal{S}' obtained by adding the product t_1 to the product set of node 2 in \mathcal{S} . The joint strategy s ceases to remain a Nash equilibrium in $\mathcal{G}(\mathcal{S}')$ since node 2 can profitably deviate to t_1 . Figure 7(b) shows an infinite improvement path starting in (t_1, t_1, t_3) . Therefore \mathcal{S} is \exists -fragile.

However, \mathcal{S} is not \mathbb{V} -fragile. First, one can check that $s = (t_1, t_3, t_3)$ is the only Nash equilibrium in $\mathcal{G}(\mathcal{S})$ for which profitable additions exist. Below we analyse the two profitable additions.

- Addition of t_1 to node 2. The following improvement path

$$2:t_1, 3:t_2, 1:t_4, 2:t_4, 3:t_4$$

starting in s terminates in the joint strategy (t_4, t_4, t_4) which is a Nash equilibrium.

- Addition of t_3 to node 1. This triggers a unique one-step improvement path that terminates in a new Nash equilibrium (t_3, t_3, t_3) . \square

5 Inefficient networks

The last two types of deficiency are concerned with product removal. These form the dual versions of the paradoxes we have seen so far. In this section we study the following notions.

We say that a social network \mathcal{S} is $\exists w$ -**inefficient** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, a contraction \mathcal{S}' of \mathcal{S} exists such that some improvement path in $\mathcal{G}(\mathcal{S}')$ leads from s to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$ such that $s' >_w s$. We note here that if the contraction was created by removing a product from the product set of node i , we impose that any improvement path in $\mathcal{G}(\mathcal{S}')$, given a starting joint strategy from $\mathcal{G}(\mathcal{S})$, begins by having node i making a choice (we allow any choice from his remaining set of products as an improvement move). Otherwise the initial payoff of node i in $\mathcal{G}(\mathcal{S}')$ is not well-defined.

As in the case of the vulnerability, we have four notions of inefficiency that correspond to the combinations XY , where $X \in \{\exists, \forall\}$ and $Y \in \{w, s\}$. For example, we say that \mathcal{S} is $\forall s$ -**inefficient** if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, a contraction \mathcal{S}' of \mathcal{S} exists such that each improvement path in $\mathcal{G}(\mathcal{S}')$ leads from s to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$ such that $s' >_s s$.

We now show that the implications between the various notions shown in Figure 2 for the case of vulnerable networks are also proper implications for inefficient networks. However, in contrast to the concept of vulnerability, here even when there are only two choices ($|P| = 2$), there exist $\forall s$ -inefficient networks.

Example 10 ($\forall s$). We exhibit in Figure 8 an example of a $\forall s$ -inefficient network. The weight of each edge is assumed to be w , and we also have the same product-independent threshold, θ , for all nodes, with $w > \theta$.

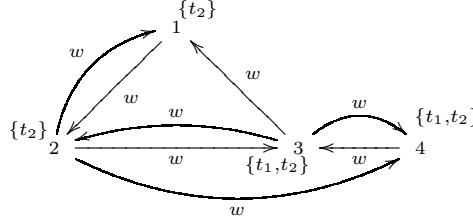


Figure 8: An example of a $\forall s$ -inefficient network

Consider as the initial Nash equilibrium the joint strategy $s = (t_2, t_2, t_1, t_1)$. It is easy to check that this is a Nash equilibrium, with a payoff equal to $w - \theta$ for all nodes. Suppose now that we remove product t_1 from the product set of node 3. We claim that all improvement paths then lead to the Nash equilibrium in which all nodes adopt t_2 .

To see this, we simply analyze all the cases that can arise. Since node 3 moves first in the improvement path, it can select either t_0 or t_2 . If it selects t_2 , then it will never change again his decision because it has a positive payoff due to node 1, and t_0 will never become a better choice. Hence it will then be the turn of node 4 to switch. If it switches to t_0 , then it will have to change again

and switch to t_2 because of the support for the choice of t_2 (from node 1). This means that eventually all nodes switch to t_2 .

Suppose now that in the beginning node 3 selected t_0 . Subsequently, node 4 can also switch to t_0 . However the main observation here is that t_0 is not a best response for any of these nodes. Both nodes receive a positive payoff for adopting t_2 thanks to a support from node 1. Hence no matter which of them switches to t_0 at the beginning of the improvement path, eventually they will both switch to t_2 .

To conclude, all improvement paths result in all nodes adopting t_2 and producing a payoff of $2w - \theta$ for each node, which is strictly better than the payoff in s . \square

Example 11 ($\forall w$). We now exhibit a network which is $\forall w$ -inefficient but not $\exists s$ -inefficient (and hence also not $\forall s$ -inefficient). We proceed as in Example 5 and add to the network given in Figure 8 a source node 5 with the product set $\{t_1\}$ and connect it to node 1, using the same weight w and threshold θ . By the same argument as in Example 5 the conclusion follows by virtue of the previous example. \square

We also remark that one can construct even simpler networks with three nodes and two products that exhibit the same behaviour.

Example 12 ($\exists s$). Next, we exhibit a network that is $\exists s$ -inefficient but not $\forall w$ -inefficient. The network is shown in Figure 9. The weight of each edge is assumed to be w , and we also have a product-independent threshold θ (with $w > \theta$), that applies to all nodes and products except three cases: $\theta(1, t_3) < \theta$, and $\theta(5, t_2) = \theta(5, t_3) > \theta$. Note that in the underlying graph, each node has exactly two incoming edges, one from the set $\{1, 2\}$ and one from $\{3, 4, 5\}$.

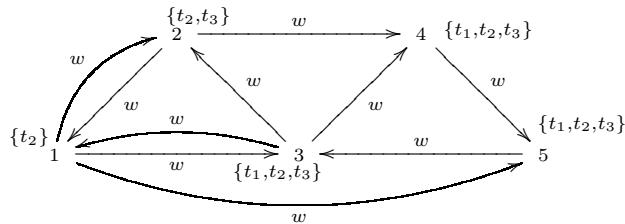


Figure 9: An example of a $\exists s$ -inefficient network

To see first that this is a $\exists s$ -inefficient network, consider the Nash equilibrium $(t_2, t_2, t_1, t_1, t_1)$. In this joint strategy each node receives support from exactly one out of its two neighbours. If we delete t_1 from the choice set of node 3, then we can see that there is an improvement path that converges to all nodes adopting t_2 (by having first node 3 adopt t_2 , followed by nodes 4 and 5). Hence in this new Nash equilibrium every node receives support from all its neighbours and the payoff of everyone has strictly increased.

To argue now that this is not a $\forall w$ -inefficient network, we need to consider all Nash equilibria and argue about all the possible contractions. One can

verify that the initial game has four Nash equilibria, namely $(t_2, t_2, t_1, t_1, t_1)$, $(t_2, t_2, t_2, t_2, t_2)$, $(t_0, t_0, t_1, t_1, t_1)$, and $(t_0, t_3, t_3, t_3, t_3)$. The idea behind this example is that in any contraction, either node 1 or node 2 or node 5 will get worse off in some improvement path. This will happen either because nodes 1 or 2 will lose support for t_2 or because node 5 will end up with product t_3 , which is a worse choice for him than t_1 .

Let us analyze the first Nash equilibrium. Note that if the contraction deletes t_2 from the product set of node 1 or 2, then node 2 will end up being worse off. If the contraction deletes t_1 from node 3, then consider the following improvement path

$$3 : t_3, 4 : t_3, 5 : t_3, 1 : t_3.$$

The last profitable deviation in this path is ensured by our assumption that $\theta(1, t_3) < \theta$. This implies that node 2 is worse off at the end. Similar improvement paths can be found if the contraction involves nodes 4 or 5.

If we consider the second Nash equilibrium, $(t_2, t_2, t_2, t_2, t_2)$, it is even easier to see that any contraction makes at least one node worse off in some improvement path since all nodes apart from node 5 receive the maximum possible payoff under this joint strategy.

Next, we analyze the third Nash equilibrium, $(t_0, t_0, t_1, t_1, t_1)$. Here the most interesting contraction is to remove t_1 from node 3 (the same things hold if we remove it from nodes 4 or 5). In that case, we can have the improvement path

$$3 : t_3, 4 : t_3, 5 : t_3, 1 : t_3$$

where node 5 becomes worse off since $\theta(5, t_3) > \theta$. The analysis for the fourth Nash equilibrium is similar and omitted. \square

Example 13 ($\exists w$). Finally, we exhibit a $\exists w$ -inefficient network that is neither $\forall w$ -inefficient nor $\exists s$ -inefficient. We proceed as in Example 11 and simply modify the previous example. We add to the network given in Figure 9 a source node 6 with the product set $\{t_2\}$ and connect it to node 1, using the same weight w and threshold θ . By the same argument as in Examples 5 and 11 the conclusion follows by virtue of the previous example.

Actually, the argument that this network is not $\forall w$ -inefficient becomes now simpler because after the addition of the source node 6 the initial game has only two Nash equilibria, namely $(t_2, t_2, t_1, t_1, t_1, t_2)$ and $(t_2, t_2, t_2, t_2, t_2, t_2)$. \square

6 Unsafe networks

Finally, we have three notions that are counterparts of the fragility notions. We say that a social network \mathcal{S} is **\exists -unsafe** (respectively, **\forall -unsafe**) if for some Nash equilibrium s in $\mathcal{G}(\mathcal{S})$, a contraction \mathcal{S}' of \mathcal{S} exists such that some (respectively, each) improvement path in $\mathcal{G}(\mathcal{S}')$ that starts in s is infinite. Further, a social network \mathcal{S} is **unsafe** if $\mathcal{G}(\mathcal{S})$ has a Nash equilibrium, while for some contraction \mathcal{S}' of \mathcal{S} , $\mathcal{G}(\mathcal{S}')$ does not.

Analogously to Section 4 each unsafe network is \forall -unsafe, while each \forall -unsafe network is \exists -unsafe. We now prove that these two implications are proper.

Example 14 (Unsafe). Let \mathcal{S}_1 be the modification of the network \mathcal{S} given in Figure 5 where node 1 and the source node marked with $\{t_1\}$ has the product set $\{t_1, t_2\}$. Consider the joint strategy in which this source node along with node 1 choose t_2 , nodes 2 and 3 choose t_3 and nodes marked by $\{t_2\}$ and $\{t_3\}$ choose the unique product in their product set. This is a Nash equilibrium in \mathcal{S}_1 . Now consider the contraction \mathcal{S}_2 of \mathcal{S}_1 in which the product t_2 is removed from the source node with product set $\{t_1, t_2\}$. Then \mathcal{S}_2 is same as the network \mathcal{S}' in Example 7. Following the argument in Example 7 we conclude that the initial network \mathcal{S}_1 is unsafe. \square

Example 15 (\forall -unsafe). Let \mathcal{S}_1 be the modification of the network \mathcal{S} given in Figure 6 where node 1 and the source node marked $\{t_1\}$ has the product set $\{t_1, t_2\}$. Consider the joint strategy in which this source node along with node 1 choose t_2 , nodes 2, 3 and the node marked $\{t_3\}$ choose t_3 and nodes marked by $\{t_2\}$ choose the unique product in their product set. This is a Nash equilibrium in \mathcal{S}_1 . Now consider the contraction \mathcal{S}_2 of \mathcal{S}_1 in which the product t_2 is removed from the source node with product set $\{t_1, t_2\}$. Then \mathcal{S}_2 is same as the network \mathcal{S}' in Example 8. Following the argument in Example 8 we conclude that the initial network \mathcal{S}_1 is \forall -unsafe but not unsafe. \square

Example 16 (\exists -unsafe). Let \mathcal{S}_1 be the modification of the network \mathcal{S} given in Figure 7 where node 2 has the product set $\{t_1, t_3, t_4\}$ and the source node marked with $\{t_3\}$ has the product set $\{t_1, t_3\}$. Consider the joint strategy in which this source node along with node 2 choose t_1 , nodes 1 and 3 choose t_2 and nodes marked by $\{t_1\}$, $\{t_2\}$ and $\{t_4\}$ choose the unique product in their product set. This is a Nash equilibrium in \mathcal{S}_1 . Now consider the contraction \mathcal{S}_2 of \mathcal{S}_1 in which the product t_1 is removed from the source node with product set $\{t_1, t_3\}$. Then \mathcal{S}_2 is same as the network \mathcal{S}' in Example 9. Following the argument in Example 9 we conclude that the initial network \mathcal{S}_1 is \exists -unsafe but not \forall -unsafe. \square

7 Networks without source nodes

Given the variety of paradoxes exhibited in the above examples it is natural to investigate the status of selected networks. In this section we focus first on networks where there are no source nodes. This is a reasonable assumption in social networks as everybody usually has some friends that influence his decisions. We later look at a special case of this class, when the underlying graph is a simple cycle. We call such networks *simple cycle networks*.

We first identify a structural property which ensures the non-existence of $\exists w$ -vulnerable networks, when the underlying graph has no source nodes. This property will imply that no form of vulnerability can arise for simple cycle networks.

Below, we only consider subgraphs that are *induced* and identify each such subgraph with its set of nodes. Recall that (V', E') is an induced subgraph of (V, E) if $V' \subseteq V$ and $E' = E \cap (V' \times V')$. For subgraphs C_1 and C_2 , we denote by $C_1 \cap C_2$ the intersection of the nodes of the graphs. We also use the following notation: for a joint strategy s and product t , $\mathcal{A}_t(s) := \{i \in V \mid s_i = t\}$, $\text{prod}(s) := \{s_i \mid i \in V\} \setminus \{t_0\}$ and denote by \bar{t} the joint strategy in which every player selects t .

We say that a (non-empty) strongly connected subgraph (in short, SCS) C of G is **self sustaining** for a product t if for all $i \in C$,

- $t \in P(i)$,
- $\sum_{j \in N(i) \cap C} w_{ji} \geq \theta(i, t)$.

So C is a self sustaining SCS for a product t if assigning this product to every node in C ensures that each node in C gets a non-negative payoff. A self sustaining SCS C is **minimal** for product t if no subgraph C' of C is a self sustaining SCS for product t . First we prove the following preparatory result.

Lemma 17. *Let $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ be a network whose underlying graph has no source nodes. If $s \neq \bar{t}_0$ is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$, then for all products $t \in \text{prod}(s)$ there exists a minimal self sustaining SCS $C \subseteq \mathcal{A}_t(s)$ for t .*

Proof. Suppose $s \neq \bar{t}_0$ is a Nash equilibrium. Take any product $t \neq t_0$ and an agent i such that $s_i = t$ (by assumption, at least one such t and i exists). Recall that $\mathcal{N}_i^t(s)$ denotes the set of neighbours of i who adopted in s the product t . Consider the set of nodes $\text{Pred} := \bigcup_{m \in \mathbb{N}} \text{Pred}_m$, where

- $\text{Pred}_0 := \{i\}$,
- $\text{Pred}_{m+1} := \text{Pred}_m \cup \bigcup_{j \in \text{Pred}_m} \mathcal{N}_j^t(s)$.

By construction for all $j \in \text{Pred}$ we have $s_j = t$ and $\mathcal{N}_j^t(s) \subseteq \text{Pred}$. Since s is a Nash equilibrium, also $\sum_{k \in \mathcal{N}_j^t(s)} w_{kj} \geq \theta(j, t)$ holds.

Consider the partial ordering $<$ between the strongly connected components of the graph induced by Pred defined by: $C < C'$ iff $j \rightarrow k$ for some $j \in C$ and $k \in C'$. Take now some SCS C' induced by a strongly connected component that is minimal in the $<$ ordering. Then for all $k \in C'$ we have $\mathcal{N}_k^t(s) \subseteq C'$ and hence $\mathcal{N}_k^t(s) \subseteq N(k) \cap C'$. This shows that C' is self sustaining. It is then straightforward to construct a minimal self sustaining SCS C for product t which is a subgraph of C' . \square

Given a network \mathcal{S} and a product $t \in \mathcal{P}$, let $\mathcal{C}_t(\mathcal{S})$ be the set of all minimal self sustaining SCSs for product t . Let $X_t(\mathcal{S}) = \bigcap_{C \in \mathcal{C}_t(\mathcal{S}), \mathcal{C}_t(\mathcal{S}) \neq \emptyset} C$ and $Y(\mathcal{S}) = \bigcap_{t \in \mathcal{P}, X_t(\mathcal{S}) \neq \emptyset} X_t(\mathcal{S})$. The following result shows that if a network \mathcal{S} does not have any self sustaining SCSs or if the intersection of the set of all minimal self sustaining SCSs is non-empty then \mathcal{S} is not $\exists w$ -vulnerable.

Theorem 18. *For a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$ whose underlying graph has no source nodes suppose that one of the following conditions holds:*

1. *for all $t \in \mathcal{P}$, $\mathcal{C}_t(\mathcal{S}) = \emptyset$,*
2. *$Y(\mathcal{S}) \neq \emptyset$.*

Then, \mathcal{S} is not $\exists w$ -vulnerable.

In particular, as we show below in the proof, condition 1 implies that $\overline{t_0}$ is the only Nash equilibrium in $\mathcal{G}(\mathcal{S})$ and condition 2 implies that $|prod(s)| = 1$ for any Nash equilibrium s in $\mathcal{G}(\mathcal{S})$ that is different from $\overline{t_0}$.

Proof. First suppose that for all $t \in \mathcal{P}$, $\mathcal{C}_t(\mathcal{S}) = \emptyset$. By Lemma 17 it follows that $\overline{t_0}$ is the only Nash equilibrium in \mathcal{S} . Consider any expansion \mathcal{S}' of \mathcal{S} . Then no player has a profitable deviation from $\overline{t_0}$ in \mathcal{S}' . Therefore, \mathcal{S} is not $\exists w$ -vulnerable.

Now suppose that $Y(\mathcal{S}) \neq \emptyset$. In this case, we first claim that every non-trivial Nash equilibrium s in \mathcal{S} has the property that $prod(s) \subseteq \{t_1\}$ for some $t_1 \in \mathcal{P}$. Suppose this is not the case then for some two different products t_1 and t_2 , $\{t_1, t_2\} \subseteq prod(s)$. Since s is a Nash equilibrium, by Lemma 17, there exists a minimal self sustaining SCS $C_1 \subseteq \mathcal{A}_{t_1}(s)$ for t_1 and a minimal self sustaining SCS $C_2 \subseteq \mathcal{A}_{t_2}(s)$ for t_2 . By definition, $\mathcal{A}_{t_1}(s) \cap \mathcal{A}_{t_2}(s) = \emptyset$ and therefore, $\mathcal{C}_{t_1}(\mathcal{S}) \cap \mathcal{C}_{t_2}(\mathcal{S}) = \emptyset$. This contradicts the assumption that $Y(\mathcal{S}) \neq \emptyset$.

Consider a Nash equilibrium s in \mathcal{S} and an expansion \mathcal{S}' . By the above claim $prod(s) \subseteq \{t_1\}$ for some $t_1 \in \mathcal{P}$. In the expansion \mathcal{S}' , if the new product t_2 is added to a node i , where $s_i = t_1$, then there is no profitable deviation from s and consequently, no improvement path starting at s in $\mathcal{G}(\mathcal{S}')$. Thus the interesting case is when the new product t_2 is added to a node i where $s_i = t_0$. We have two cases.

- *Case 1: $t_2 \neq t_1$. Since $prod(s) \subseteq \{t_1\}$, there is no profitable deviation from s and therefore, no improvement path starting at s in $\mathcal{G}(\mathcal{S}')$.*
- *Case 2: $t_2 = t_1$. Consider any improvement path in $\mathcal{G}(\mathcal{S}')$ that leads to a Nash equilibrium s' in $\mathcal{G}(\mathcal{S}')$. Since s is a Nash equilibrium in $\mathcal{G}(\mathcal{S})$ the first profitable deviation in the improvement path is of the form $i : t_1$. In the improvement path, if a joint strategy s^2 is obtained from s^1 by having some nodes switch to product t_1 and t_1 is a best response for a node j to s_{-j}^1 , then t_1 is also a best response for j to s_{-j}^2 . Indeed, by the join the crowd property $p_j(t_1, s_{-j}^2) \geq p_j(t_1, s_{-j}^1) \geq p_j(t_1, s_{-j}^1) = 0 = p_j(t_0, s_{-j}^2)$. So the only deviations in this improvement path are to t_1 . Consequently in s' which is a Nash equilibrium, t_1 is the product selected by node i (i.e., $s'_i = t_1$) and $p_i(s') > p_i(s)$. Therefore, the network is not $\exists w$ -vulnerable.*

□

7.1 Simple cycle networks

In this subsection we focus only on networks where the underlying graph is a simple cycle. We start with the following corollary to Theorem 18.

Corollary 19. *Simple cycle networks are not $\exists w$ -vulnerable (and a fortiori not XY -vulnerable, where $X \in \{\exists, \forall\}$ and $Y \in \{w, s\}$).*

Proof. Consider a network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, where G is a simple cycle. First note that each non-empty self sustaining SCS for a product t equals G and is minimal.

Suppose now that there exists $t \in \mathcal{P}$ such that $\mathcal{C}_t(\mathcal{S}) \neq \emptyset$. Then the above observation implies that $Y(\mathcal{S}) \neq \emptyset$. By Theorem 18 it follows that \mathcal{S} is not $\exists w$ -vulnerable. \square

The remaining types of deficiency are easy to determine. For the case of fragile networks we prove the following result.

Theorem 20. *Simple cycle networks are not $\exists w$ -fragile (and a fortiori not $\forall w$ -fragile and not fragile).*

Proof. Consider a simple cycle network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, a Nash equilibrium s of $\mathcal{G}(\mathcal{S})$, and an expansion \mathcal{S}' of \mathcal{S} . By Theorem 7 in [16] s is of the form \bar{t} , where $t \in \mathcal{P} \cup \{t_0\}$. Hence s remains a Nash equilibrium of $\mathcal{G}(\mathcal{S}')$. Consequently \mathcal{S} is not $\exists w$ -fragile. \square

In the case of inefficient networks we have the following result.

Theorem 21.

- (i) *There exists a simple cycle network \mathcal{S} that is $\exists s$ -inefficient (and a fortiori $\exists w$ -inefficient).*
- (ii) *Simple cycle networks are not $\forall w$ -inefficient (and a fortiori not $\forall s$ -inefficient).*

Proof. (i) Consider the network shown in Figure 10. Suppose that $\theta(i, t_1) > \theta(i, t_2)$ for all nodes $i = 1, 2, 3$ and that $s = \bar{t}_1$ is a Nash equilibrium. Starting from s , suppose we remove t_1 from the product set of node 1. Then there exists a finite improvement path where all the nodes end up adopting t_2 , by simply having node 1 adopt t_2 and then having the remaining nodes follow their best response. Since $\theta(i, t_1) > \theta(i, t_2)$, all nodes are strictly better off in this new Nash equilibrium, \bar{t}_2 .

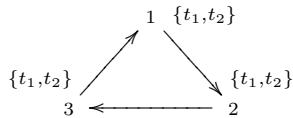


Figure 10: An $\exists s$ -inefficient simple cycle network

(ii) Consider a simple cycle network $\mathcal{S} = (G, \mathcal{P}, P, \theta)$, a Nash equilibrium s of $\mathcal{G}(\mathcal{S})$, and a contraction \mathcal{S}' of \mathcal{S} . By Theorem 7 in [16], s is of the form \bar{t} , where $t \in \mathcal{P} \cup \{t_0\}$. If $s = \bar{t}_0$, then it is impossible that by deleting any product we could make some player better off. Suppose $s = \bar{t}_1$ for some product t_1 . If we delete t_1 from some product set, say of node 1, then there is always an improvement path that terminates at the Nash equilibrium \bar{t}_0 (simply start with node 1 adopting t_0 , and proceed clockwise. Then gradually every other node will switch to t_0 since they eventually lose support for t_1). Hence no node is better off in this new Nash equilibrium. In conclusion, there can be no Nash equilibrium from which all improvement paths after the contraction will make the set of nodes weakly better off. \square

Finally, we consider the case of unsafe networks.

Theorem 22.

- (i) *There exists a simple cycle network \mathcal{S} that is \exists -unsafe.*
- (ii) *Simple cycle networks are not \forall -unsafe (and a fortiori not unsafe).*

Proof. (i) Consider the network shown in Figure 11(a) and assume that \bar{t} is a Nash equilibrium.

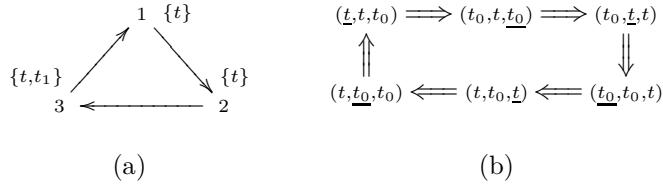


Figure 11: A simple cycle network and an infinite improvement path

By removing from the product set of node 3 the product t we get in the resulting game an infinite improvement path depicted in Figure 11(b). (In each joint strategy we underline the strategy that is not a best response to the choice of other players.) So the initial network is \exists -unsafe.

(ii) By Theorem 28 in [16] for every simple cycle network \mathcal{S} there exists a finite improvement path in $\mathcal{G}(\mathcal{S})$. This implies both claims. \square

The above analysis does not carry through to all strongly connected graphs. Indeed, we showed in particular that simple cycle networks cannot be $\forall w$ -vulnerable and also not $\forall s$ -inefficient. However, in Example 3 we exhibited a network that is $\forall w$ -vulnerable and in Example 10 a network that is $\forall s$ -inefficient. The underlying graphs of both networks are strongly connected.

8 Conclusions

In this paper we provided a systematic study of paradoxes that can arise in social networks with multiple products. To analyze them we used a natural game-theoretic framework in the form of social network games recently introduced in [15] and [16]. These games do not need to have (pure) Nash equilibria. As a result more types of paradoxes can arise than in the case of congestion games with its celebrated Braess paradox.

These paradoxes allow us to better understand possible undesired consequences of modifying the sets of products available to the agents forming a social network. The focus of the paper was on identifying these paradoxes and on determining their relative strength. One problem remained open: do $\forall s$ -vulnerable networks exist?

In future work we plan to assess the computational complexity of determining the presence of these paradoxes and plan to analyze other selected networks. We also plan to expand our analysis of selected classes of networks, determining which paradoxes can then be present.

Finally, in our analysis we assumed that the agents can refrain from selecting a product. We also plan to consider an alternative version of the social networks in which each agent has to choose a product. This corresponds to natural situations, for instance when pupils have to choose a primary school or when each student has to select a laptop. Such social networks can be studied using the social network games in which the strategy t_0 is not available. We have already noticed that this change leads to a different analysis.

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